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OPTIMAL CORRECTION PROBLEM OF
A MULTIDIMENSIONAL STOCHASTIC SYSTEM

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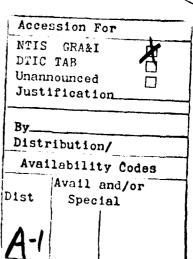Department of Mathematics
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OPTIMAL CORRECTION PROBLEM OF
A MULTIDIMENSIONAL STOCHASTIC SYSTEM

by

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ABSTRACT

We consider a stochastic dynamic system which is governed by a multidimensional diffusion process with constant drift and diffusion coefficients. The correction corresponds to an additive input which is under control. There is no limit on the rate of input into the system. The objective is to minimize the expected cumulative cost associated with the position of the system and the amount of control excerted.

It is proved that Hamilton-Jacobi-Bellman's equation of the problem has a solution, which corresponds to the optimal cost of the problem. An existence of optimal policy is proved.

---

1. INTRODUCTION

This problem is motivated by studies of a dissipative system under uncertainty. A typical model would be an automatic cruise control of an aircraft subject ot uncertain wind conditions. The problem is to balance costs associated with deviation of the airplane from the prescribed course and fuel expenditures resulting from the correction of the course.

We assume that in absence of control the fluctuations of our stochastic system are described by a multidimensional Brownian motion with constant n-dimensional vector - drift g and $n \times n$ diffusion matrix σ.

$$y_x(t) = x + gt + \sigma w(t)$$

Here x is the initial position, and $w(\cdot)$ is a n-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

The "quality" of the position of the system is measured by a function h. We assume that h is a strictly convex nonnegative function such that

$$h(x)/|x| \to \infty \quad \text{as} \quad |x| \to \infty.$$

The control is realized by $2n$ increasing, $\mathcal{F}_t$-adapted processes $\nu_i^+(t), \nu_i^-(t), i = 1, 2, \ldots, n$. The control functional $\nu(t)$ is a n-dimensional $\mathcal{F}_t$-adapted process of bounded variation defined

$$\nu(t) = (\nu_1(t), \nu_2(t), \ldots, \nu_n(t)), \tag{1.1}$$

$$\nu_i(t) = \nu_i^+(t) - \nu_i^-(t). \tag{1.2}$$

The dynamics of the system under control is then

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$$y_x(t) = x + gt + \sigma w(t) + \nu(t). \tag{1.3}$$

With each initial position x and each control functional ν we associate a cost

$$J_x(\nu) = E\Big\{ \int_0^\infty e^{-\alpha t} h(y_x(t)) dt +$$
$$\sum_{i=1}^n \Big[a_i \int_0^\infty e^{-\alpha t} d\nu_i^+(t) + b_i \int_0^\infty e^{-\alpha t} d\nu_i^-(t) \Big] \Big\}, \tag{1.4}$$

where a_i and b_i, $i = 1,2,\ldots,n$ are positive constanst and $\alpha > 0$ is a discount factor. Denote by V the set of all n-dimensional $\mathcal{F}_t$-adapted processed ν represented in the form (1.1), (1.2). We are looking for

$$u(x) = \inf\{J_x(\nu) : \nu \; \epsilon \; V\} \tag{1.5}$$

and ν^* such that

$$u(x) = J_x(\nu^*). \tag{1.6}$$

Let

$$[a_{ij}] = \sigma\sigma^* \tag{1.7}$$

and $\nabla = \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\big)$. Put

$$A = -\frac{1}{2} \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^n g_i \frac{\partial}{\partial x_i} + \alpha \equiv$$
$$-\frac{1}{2} tr(\sigma\sigma^* \nabla^2) - g \nabla + \alpha. \tag{1.8}$$

For $q = (q_1, q_2, \ldots, q_n) \; \epsilon \mathbf{R}^n$ let $\|q\| = \max |q_i|$, Put

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$$\beta(q) = \max_{\substack{\xi, \eta \in \mathbf{R}^n; \xi, \eta \geq 0 \\ \|\xi\|, \|\eta\| \leq 1}} \sum_{i=1}^{n} [-\xi_i(a_i + q_i) - \eta_i(b_i - q_i)] \qquad (1.9)$$

Note that

$$\beta(q) \geq 0 \text{ and } \beta(q) = 0 \text{ iff } -a_i \leq q_i \leq b_i, i = 1, 2, \dots, n. \qquad (1.10)$$

We will show that the optimal cost function u given by (1.5) satisfies that following Hamilton-Jacobi-Bellman equation

$$\max(Au - h, \gamma(\nabla u) - 1) = 0, \qquad (1.11)$$

where $\gamma(q) = \max_{1 \leq i \leq n}[(q_i^+/a_i) \vee (q_i^-/b_i)]$, and we prove the existence of the optimal cost.

In section 2 we consider a family of problems in which the allowable controls are absolutely continuous with the rates uniformly bounded. We derive estimates for the cost functions of these problems. In section 3 it is shown that a subsequence of cost functions for absolutely continuous control problems converges to $u(x)$ given by (1.5). Section 4 is devoted to construction of the optimal policy.

2. Absolutely continuous control problems.

Here and in sequel we assume that the function h is strictly convex and roughly speaking is of polynomal growth. More precisely there exist $p > 1$ and constants C_0, C_1 and C_2 such that for any $0 < \lambda < 1$, any $x \in \mathbf{R}^n$ and any x' such that $|x'| \leq 1$,

$$0 \leq h(x) \leq C_0(1 + |x|)^p, \qquad (2.1)$$

$$|h(x) - h(x + x')| \leq C_1(1 + h(x) + h(x + x'))^{1-p^{-1}}|x'|, \qquad (2.2)$$

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$$0 < h(x + \lambda x') + h(x - \lambda x') - 2h(x) \leq C_2 \lambda^2 (1 + h(x))^q, \quad q = (1 - \frac{2}{p})^+. \tag{2.3}$$

Let V_ϵ be the set of all $\nu \ \epsilon \ V$ such that

$$\nu(t) = \int_0^t \dot{\nu}(s) ds, \quad |\dot{\nu}(s)| < \epsilon^{-1} \quad \text{for all} \quad s > 0, \tag{2.4}$$

and

$$u_\epsilon(x) = \inf_{\nu \epsilon V_\epsilon} J_x(\nu) \tag{2.5}$$

Formal application of the dynamic programming principle yields the following equation

$$Au_\epsilon + \epsilon^{-1} \beta(\nabla u_\epsilon) = h \tag{2.6}$$

(see Fleming and Rishel (1975)). The next theorem establishes the properties of u_ϵ which will be used in sequel.

(2.7) <u>Theorem.</u> Suppose $h(x)$ satisfies (2.1) - (2.3). Then there exist $\hat{C}_0, \hat{C}_1, \hat{C}_2$ independent of $\epsilon \epsilon (0, 1]$ such that for each $\lambda \epsilon (0, 1)$ and each x' with $|x'| \leq 1$ the function $u_\epsilon(x)$ satisfies (2.8)-(2.10) below

$$0 \leq u_\epsilon(x) \leq \hat{C}_0 (1 + |x|)^p, \tag{2.8}$$

$$|u_\epsilon(x) - u_\epsilon(x + x')| \leq \hat{C}_1 (1 + |x| + |x + x'|)^{p-1} |x'|, \tag{2.9}$$

$$0 \leq u_\epsilon(x + \lambda x') + u_\epsilon(x - \lambda x') - 2u_\epsilon(x) \leq \hat{C}_2 \lambda^2 (1 + |x|)^{(p-2)^+}. \tag{2.10}$$

Moreover (2.8)-(2.10) is also true for u.

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<u>Proof</u> 1°. Putting $\nu \equiv 0$, we get from (2.5)

$$u_\epsilon(x) \le J_x(0) = E\Big\{ \int_0^\infty e^{-\alpha t} h(x + \sigma w(t) + gt) dt \Big\}$$

$$\le E\Big\{ \int_0^\infty e^{-\alpha t} C_0 (1 + |x + \sigma w(t) + gt|^p) dt \Big\}$$

$$\le C_0 \alpha^{-1} + C_0 E\Big\{ \int_0^\infty e^{-\alpha t} |x + \sigma w(t) + gt|^p) dt \Big\}$$

$$\le C_0 \alpha^{-1} + C_0 \alpha^{-1} c |x|^p,$$

where c is a constant dependent on α, σ and g. Putting $\hat{C}_0 = C_0 \alpha^{-1} \max(1, c)$, we get (2.8).

2°. Consider

$$u_\epsilon(x) - u_\epsilon(x + x') = \inf_\nu \sup_{\nu'} J_x(\nu) - J_{x+x'}(\nu')$$

$$\le \sup_{\nu'} J_{x+x'}(\nu') - J_x(\nu') = \sup_\nu J_x(\nu) - J_{x+x'}(\nu).$$

Likewise $u_\epsilon(x + x') - u_\epsilon(x) \le \sup_\nu J_{x+x'}(\nu) - J_x(\nu)$. Therefore

$$|u_\epsilon(x) - u_\epsilon(x + x')| \le \sup_\nu |J_x(\nu) - J_{x+x'}(\nu)|$$

$$\le \sup_\nu E\Big\{ \int_0^\infty e^{-\alpha t} |h(y_x(t)) - h(y_{x+x'}(t))| dt \Big\}$$

$$\le \sup_\nu C_1 E\Big\{ \int_0^\infty |1 + h(y_x(t)) - h(y_{x+x'}(t))|^{1-1/p}$$

$$|(y_x(t) - (y_{x+x'}(t)| e^{-\alpha t} dt \Big\}$$

$$\le C_1 |x'| \alpha^{-1/p} \sup_\nu (E\Big\{ \int_0^\infty |1 + h(y_x(t))$$

$$+ h(y_{x+x'}(t))| e^{-\alpha t} dt \Big\})^{1-1/p}.$$

(2.11)

The last inequality in (2.11) is due Hôlder's inequality. By virtue of (2.8), we can consider

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only those in (2.11) ν for which $E \int_0^\infty h(y_x(t))e^{-\alpha t}dt \leq C_0(1 + |x|)^p$. Therefore, applying Hölder's inequality to the last line of (2.11) once more,

$$|u_\epsilon(x) - u_\epsilon(x + x')| \leq |x'|C_1\alpha^{-1}(2 + C_0/\alpha)(1 + |x| + |x + x'|)^{p-1},$$

whereas (2.9) follows.

3°. Since h is a convex function, the function $J_x(\nu)$ is convex in (x, ν). Because V_ϵ is a convex set the function $u_\epsilon(x)$ is convex as well. Therefore the first part of the inequality (2.10) follows. Put $x_1 = x + \lambda x', x_2 = x - \lambda x'$

$$
\begin{aligned}
u_\epsilon(x &+ \lambda x') + u_\epsilon(x - \lambda x') - 2u_\epsilon(x) \\
&= \inf_{\nu_1} \inf_{\nu_2} \sup_\nu \{J_{x_1}(\nu_1) + J_{x_2}(\nu_2) - 2J_x(\nu)\} \\
&\leq \sup_\nu J_{x_1}(\nu) + J_{x_2}(\nu) - 2J_x(\nu) \\
&\leq E\{\int_0^\infty e^{-\alpha t}C_2\lambda^2(1 + h(y_x(t)))^q dt\}.
\end{aligned}
\tag{2.12}
$$

If $q = 0$ then (2.12) implies (2.10) in an obvious manner. If $q > 0$ (that is $p > 2$) then by Hölder's inequality

$$
\begin{aligned}
E\Big\{ \int_0^\infty &e^{-\alpha t}(1 + h(y_x(t)))^q dt \Big\} \\
&\leq \Big(\int_0^\infty e^{-\alpha t}dt \Big)^{2/p} \Big(E\Big\{ \int_0^\infty e^{-\alpha t}[1 + h(y_x(t))]\Big\} \Big)^{(p-2)/p} \\
&\leq \alpha^{-2/p}(\alpha^{-q} + \hat{C}_0(1 + |x|)^p)^{(p-2)/p}
\end{aligned}
\tag{2.13}
$$

The last inequality in (2.13) is due to (2.8). Simple analysis show that (2.13) implies (2.10)

The proof of (2.8)-(2.10) for u is the same.

(2.14) <u>Theorem</u>. The optimal cost $u_\epsilon(x)$ is the unique solution of (2.6) under the conditions (2.8)-(2.10).

<u>Proof 1°.</u> For a nondegenerate σ the existence of a solution follows from the classical results (see Fleming and Rishel (1975)). If σ is degenerate then consider $\sigma_\delta = [\sigma, \delta I]$ which is a $2n \times n$ matrix. Note that

$$\sigma_\delta \sigma_\delta^* = \sigma\sigma^* + \delta^2 I$$

is nondegenerate for all sufficiently small δ. Consider a new optimization problem in which w in (1.3) is a 2n-dimensional standard Brownian motion while σ is replaced by σ_δ. Let $u_{\epsilon,\delta}$ be the corresponding optimal cost given by (2.5). Then $u_{\epsilon,\delta}$ satisfies

$$-\frac{\delta^2}{2} \nabla u_{\epsilon,\delta} + A u_{\epsilon,\delta} + \epsilon^{-1}\beta(\nabla u_{\epsilon,\delta}) = h \qquad (2.14)$$

Repeating step by step the proof of the Theorem (2.7), we can see that (2.8)-(2.10) hold with $\hat{C}_0, \hat{C}_1, \hat{C}_2$ independent of $\delta > 0$ for all sufficiently small δ. If follows that $u_{\epsilon,\delta}(x), \nabla u_{\epsilon,\delta}(x)$ and $A u_{\epsilon,\delta}(x)$ is locally uniformly (in δ) bounded in x. The latter implies existence of a subsequence $\delta_k \to 0$ such that $|\triangle u_{\epsilon,\delta_k}(x)|$ is locally bounded in x and

$$u_{\epsilon,\delta_k}(x) \to u_\epsilon(x),$$
$$\nabla u_{\epsilon,\delta_k}(x) \to \nabla u_\epsilon(x),$$

locally uniformly in x and $A u_{\epsilon,\delta_k} \to A u_\epsilon$ as a distribution (in Schwartz' sense). Passing to a limit in (2.14), we get the validity of (2.6) for u_ϵ.

2°. To prove uniqueness assume that there are two solutions u_ϵ and v_ϵ of (2.6). Let

$$A_0 = -\frac{1}{2}\sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_i g_i \frac{\partial}{\partial x_i}.$$

(Recall $A = A_0 + \alpha I$). Then

$$A_0(u_\epsilon - v_\epsilon) = -\alpha(u_\epsilon - v_\epsilon) + \epsilon^{-1}(\beta(\nabla v_\epsilon) - \beta(\nabla u_\epsilon)). \qquad (2.15)$$

Let $W(x) = u_\epsilon(x) - v_\epsilon(x)$ and $w(x) = W(x)\psi(x, \lambda)$, where $\psi(x, \lambda) = (\lambda + |x|^2)^{-p}$ (the value for the constant λ will be chosen later). If follows from (2.8) that $w(x) \to 0$ as $|x| \to \infty$. Suppose $w(x) \not\equiv 0$ and $w(x) > 0$ for some x. (If $w(x) < 0$ then consider $v_\epsilon - u_\epsilon$). Let

$$w(x_0) = \max_{x \in \mathbf{R}^n} w(x) > 0. \tag{2.16}$$

Calculations show that

$$A_0 \psi(x, \lambda) + [tr(\sigma\sigma^* \bigtriangledown \psi(x, \lambda) \bigtriangledown \psi(x, \lambda)^*)]/\psi(x, \lambda) = \delta(x, \lambda)\psi(x, \lambda), \tag{2.17}$$

where $\sup_x |\delta(x, \lambda)| \to 0$ if $\lambda \to \infty$, and

$$\beta(\bigtriangledown v_\epsilon) - \beta(\bigtriangledown u_\epsilon) = \gamma(x, \epsilon)[\bigtriangledown v_\epsilon - \bigtriangledown u_\epsilon] \equiv -\gamma(x, \epsilon) \bigtriangledown W(x), \tag{2.18}$$

where $\|\gamma(x, \epsilon)\| \le 1$. In view of (2.16), $\bigtriangledown w(x_0) = 0$. Therefore

$$\bigtriangledown W(x_0)\psi(x_0, \lambda) = -W(x_0) \bigtriangledown \psi(x_0, \lambda). \tag{2.19}$$

and

$$\bigtriangledown W(x_0) = -W(x_0) \bigtriangledown \psi(x_0, \lambda)/\psi(x_0, \lambda). \tag{2.20}$$

Since $\bigtriangledown \psi(x_0, \lambda)/\psi(x_0, \lambda) \to 0$ as $\lambda \to \infty$, we can combine (2.20) and (2.19) to get

$$\beta(\bigtriangledown v_\epsilon(x_0)) - \beta(\bigtriangledown u_\epsilon(x_0)) = W(x_0)\hat{\delta}(\epsilon, \lambda), \tag{2.21}$$

where $\sup_{0 < \epsilon \le 1} |\hat{\delta}(\epsilon, \lambda)| \to 0$ as $\lambda \to \infty$. Also,

$$A_0 w(x_0) = A_0 W(x_0)\psi(x_0, \lambda) + W(x_0)A_0\psi(x_0, \lambda) - tr(\sigma\sigma^* \bigtriangledown W(x_0) \bigtriangledown \psi(x_0, \lambda)). \tag{2.22}$$

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Applying to the first term in the right hand side of (2.22) and using
(2.21), (2.17) and (2.19), we get

$$A_0 w(x_0) = [-\alpha W(x_0) + \epsilon^{-1} W'(x_0)\hat{\delta}(\epsilon,\lambda))\psi(x_0,\lambda) \cdot W(x_0,\lambda)$$

$$-tr(\sigma\sigma^*(-\nabla\psi(x_0,\lambda)W'(x_0) \cdots \psi(x_0,\lambda))\psi(x_0,\lambda)$$

$$= W(x_0)\psi(x_0,\lambda)(-\alpha + \epsilon^{-1}\hat{\delta}(\epsilon,\lambda) + \delta(x_0,\lambda))$$

Now choose $\lambda > 0$ such that $-\alpha + \epsilon^{-1}\hat{\delta}(\epsilon,\lambda) + \delta(x_0,\lambda) < 0$. In view of (2.16) (3.3)
this implies $w(x_0) \leq 0$ which is in contradiction with (2.16).

3. Solution of the Hamilton-Jacobi-Bellman equation

Passing to a limit in (2.6), as $\epsilon \to 0$ (and assuming for a moment convergence of u_ϵ, ∇u_ϵ
and Au_ϵ to $u, \nabla u$ and Au respectively), we get inequalities

$$Au(x) \leq h, \tag{3.1}$$

$$-a_i \leq \nabla u_i(x) \leq b_i. \tag{3.2}$$

Assuming also that at each point x at least one of (3.1) and (3.2) is tight, we get (1.11).
In this section we will show that u given by (1.5) is actually a solution of (1.11).

(3.3) <u>Theorem.</u> There exist $c_1, c_2 > 0$ such that

$$c_1|x| \leq u(x) \leq c_2(1 + |x|). \tag{3.4}$$

<u>Proof</u> 1°. For $x \epsilon R^n$ put

$$\eta(x) = \begin{cases} x/|x| - x, & \text{if } |x| > 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let

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$$\hat{\nu}_x(t) = \eta(x)1_{|x|>2} + \sum_{\tau_k \le t} \eta(x_{\tau_k}), \tag{3.5}$$

where

$$\tau_0 = 0,$$

$$\tau_k = \inf\{t > \tau_{k-1} : |x + gt + \sigma w(t) + \sum_{n=1}^{k-1} \tau_n| = 2\}.$$

The policy $\hat{\nu}_x(t)$ acts in the following manner. When the process is outside the ball of radius 2 it is instantaneously moved inside the ball of radius 1. Then there is no action until the process reaches the boundary of the ball of radius 2, at which moment it is moved again into the ball of radius 1 and so on. Let

$$U(x) = E\Big\{\int_0^\infty e^{-\alpha t}h(x + gt + \sigma w(t) + \hat{\nu}_x(t))dt$$

$$+ \sum_{k=1}^\infty e^{-\alpha \tau_k} \sum_{i=1}^n \Big[a_i(\hat{\nu}_{xi}(\tau_k) - \hat{\nu}_{xi}(\tau_k^-))^+ +$$

$$b_i(\hat{\nu}_{xi}(\tau_k) - \hat{\nu}_{xi}(\tau_k^-))^-\Big]\Big\}, \quad \text{if} \quad |x| < 2.$$

Then

$$J_x(\hat{\nu}_x) = \begin{cases} U(x) & \text{if } |x| < 2, \\ U(x/|x|) + \sum_{i=1}^n (a_i(x/|x| - x)_i^+ + b_i(x/|x|_i - x)_i^-), & \text{if } |x| \ge 2. \end{cases} \tag{3.6}$$

It is easy to see that $U(x)$ is a continuous function and therefore bounded in $\{x : |x| < 2\}$. Thus formula (3.6) implies that for $\hat{\nu}_x$ given by (3.5), $J_x(\hat{\nu}_x)/|x|$ is bounded, whereas the second inequality in (3.4) follows.

2°. Let $a = \min(a_i \wedge b_i, i = 1, 2, \ldots, n)$. Let

$$\mathcal{A}_R = \{|gt + \sigma w(t)| < R \text{ for all } 0 \le t \le 1\}$$

Choose R such that

$$P(\mathcal{A}_R) > \frac{1}{2},$$

and

$$h(x)/|x| > a \quad \text{for all} \quad |x| > R. \tag{3.7}$$

Let $|x| > 3R$ and ν be any policy.

Put $B = \{|y_x(t)| \equiv |x + gt + \sigma w(t) + \nu(t)| > |x|/3 \text{ for all } t \le 1\}$. Then

$$J_x(\nu) \ge e^{-1} E\left\{\int_0^1 h(y_x(t))dt + a\sum_{i=1}^n [\nu_i^+(1) + \nu_i^-(1)]\right\} \ge$$

$$e^{-1} E\left\{\int_0^1 h(y_x(t))1_B dt + a\sum_{i=1}^n [\nu_i^+(1) + \nu_i^-(1)](1 - 1_B))1_{\mathcal{A}_R}\right\}. \tag{3.8}$$

We have $1_B h(y_x(t)) > 1_B a|x|/3$ by virtue of (3.7). On $\mathcal{A}_R$ the quantity $|x + gt + \sigma w(t)|$ exceeds $2|x|/3$ for all $t \le 1$. Therefore on $\bar{B} \cap \mathcal{A}_R$

$$\sum_{i=1}^n (\nu_i^+(1) + \nu_i^-(1)) > |x|/3.$$

Hence (3.8) exceeds

$$e^{-1} E\left\{\int_0^1 (a|x|/3)1_B dt + (a|x|/3)(1 - 1_B))1_{\mathcal{A}_R}\right\} \ge e^{-1}(a|x|/3)P(\mathcal{A}_R) \ge e^{-1}a|x|/6 \tag{3.9}$$

Inequality (3.9) implies the first inequality in (3.4).

(3.10) <u>Theorem</u>. There exists a constant c and a sequence $\varepsilon_k \downarrow 0$ such that for any $R > 0$ there exists N such that for every $k > N$

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$$u_{\varepsilon_k}(x) < c(1 + |x|) \quad \text{for all} \quad |x| < R.$$

Proof. Consider a space $\mathcal{V}$

$$\mathcal{V} = \{v : v \in L_\psi^2, |\nabla v| \in L_{\tilde\psi}^2\},$$

where $\psi = (\lambda + |x|^2)^{-p-n}, \tilde\psi = (\lambda + |x|^2)^{-p-n+1}$ and L_ψ^2 denote the set of functions v on $\mathbf{R}^n$ for which $v^2 \psi$ is integrable.

Inequalities (2.8)-(2.10) show that $u_\varepsilon(x), \nabla u_\varepsilon(x)$ and $\|\frac{\partial^2 u_\varepsilon(x)}{\partial x^2}\|$ are uniformly bounded in ε on every compact subset of $\mathbf{R}^n$. Also the same inequalities show that u_ε is uniformly bounded in $L_\psi^2, |\nabla u_\varepsilon|$ is uniformly bounded in $L_{\tilde\psi}^2$ and $\|\frac{\partial^2 u_\varepsilon(x)}{\partial x^2}\|$ is uniformly bounded in $L_{\tilde\psi}^2$. Hence there exists a function $u_0 \in \mathcal{V}$ and a subsequence ε_k such that

$$u_{\varepsilon_k} \to u_0 \quad \text{weakly in} \quad \mathcal{V}, \tag{3.11}$$

$$Au_{\varepsilon_k} \to Au_0 \quad \text{weakly in} \quad L_\psi^2 \tag{3.12}$$

and $\varepsilon_k^{-1}\beta(\nabla u_{\varepsilon_k}) = h - Au_{\varepsilon_k}$ is bounded in L_ψ^2. Therefore

$$\lim_{k \to \infty} \beta(\nabla u_{\varepsilon_k}) = 0. \tag{3.13}$$

Since convergence of ∇u_{ε_k} is locally uniform, by virtue of (1.10), for very R there exists N such that for every $k > N$

$$-a_i - \delta < \nabla u_{\varepsilon_k i}(x) < b_i + \delta \quad \text{for all} \quad |x| < R. \tag{3.14}$$

Since $u_{\varepsilon_k}(x)$ is monotonically decreasing, (3.14) implies the statement of the theorem.

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(3.15) Proposition. Let V_0 denote the set of $\nu \, \epsilon \, V$ such that $\nu(0) = 0$. then

$$u(x) = \inf_{\nu \epsilon V_0} J_x(\nu).$$

Proof $1°$. We may consider only those ν for which $J_x(\nu)$ is finite. First show that in the mimimization problem (1.5) we can consider only those ν for which there exists r (possibly dependent on x) such that

$$|\nu(0)| < r \quad \text{a.s.} \, . \tag{3.16}$$

Let $\hat{\nu}_x$ be a policy given by (3.5) for which $J_x(\hat{\nu}_x) < c_2(1 + |x|)$. For any policy ν and initial state x consider a policy

$$\nu^r(t) = \begin{cases} \nu(t) & \text{if } |\nu(0)| < r, \\ \hat{\nu}_x(t) & \text{if } |\nu(0)| \geq r. \end{cases}$$

If $|r| > |x|$ then $|\nu^r(0)| < r$. Using the first inequality in (3.4),

$$J_x(\nu) - J_x(\nu^r) \geq E\{c_1|y_x(0)| - c_2(1 + |x|); |\nu(0)| \geq r\}$$
$$\geq (c_1(r - |x|) - c_2(1 + |x|))P(|\nu(0)| \geq r\}. \tag{3.17}$$

If $r > (1 + |x|)c_2/c_1 + |x|$, then $J_x(\nu^r) < J_x(\nu)$ while $|\nu^r(0)| < r$.

Likewise, we can show that every policy is dominated by the one for which for every stopping time τ

$$|\nu(y_x(\tau)) - \nu(y_x(\tau-))| \leq (1 + |y_x(\tau-)|)c_2/c_1 + |y_x(\tau-)|. \tag{3.18}$$

$2°$. Let ν be any policy subject to (3.16) and (3.18) and let

$$\tilde{\nu}(t;\varepsilon) = \begin{cases} \nu(t) & \text{if } t \geq \varepsilon, \\ \nu(0)t/\varepsilon + \nu(t)(t - \varepsilon)/\varepsilon & \text{if } t < \varepsilon \, . \end{cases}$$

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Let $\tilde{y}_x$ be the trajectory corresponding to $\tilde{\nu}(t;\delta)$ and

$$\tau = \inf\{t : |\nu(t)| > r+1 \quad \text{or} \quad |\tilde{y}_x(t)| > |x|+r+1\}$$

$$\nu(t;\varepsilon) = \begin{cases} \tilde{\nu}(t;\varepsilon), & \text{if } \tau > \varepsilon, \quad \text{or} \quad \text{if} \quad \tau < \varepsilon \quad \text{and} \quad t \le \tau, \\ \hat{\nu}_{\tilde{y}_x(\tau)}(t-\tau) + \tilde{\nu}(\tau;\varepsilon), & \text{if } \tau \le \varepsilon \quad \text{and} \quad t > \tau, \end{cases}$$

where $\hat{\nu}$ is given by (3.5). Let $y_x(t)$ be the trajectory corresponding to ν and $y_x(t;\varepsilon)$ be the trajectory corresponding to $\nu(\cdot;\varepsilon)$. Then, $\nu(t,\varepsilon) = \nu(t)$ for all $t > \varepsilon$ on $\{\tau > \varepsilon\}$. Also, in view of (3.16)

$$P\{\tau < \varepsilon\} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.19}$$

Consider

$$|J_x(\nu) - J_x(\nu(\cdot;\varepsilon))| \le E\left\{\int_0^\varepsilon h(y_x(t))e^{-\alpha t}dt; \tau > \varepsilon\right\}$$

$$+ E\left\{\int_0^\varepsilon h(y_x(t;\varepsilon))e^{-\alpha t}dt; \tau > \varepsilon\right\} + E\left\{\int_0^\infty h(y_x(t))e^{-\alpha t}dt\right.$$

$$+ \sum_{i=1}^n \left[a_i \int_0^\infty e^{-\alpha t}d\nu_i^+(t) + b_i \int_0^\infty e^{-\alpha t}d\nu_i^-(t)\right]; \tau \le \varepsilon\right\} + E\left\{\int_0^\infty h(y_x(t;\varepsilon))e^{-\alpha t}dt\right.$$

$$+ \sum_{i=1}^n \left[a_i \int_0^\infty e^{-\alpha t}d\nu_i^+(t;\varepsilon) + b_i \int_0^\infty e^{-\alpha t}d\nu_i^-(t;\varepsilon)\right]; \tau \le \varepsilon\right\}$$

$$+ \left|E\left\{\sum_{i=1}^n a_i\left(\int_0^\varepsilon e^{-\alpha t}d\nu_i^+(t) - \int_0^\varepsilon e^{-\alpha t}d\nu_i^+(t;\varepsilon)\right); \tau > \varepsilon\right\}\right|$$

$$+ \left|E\left\{\sum_{i=1}^n b_i\left(\int_0^\varepsilon e^{-\alpha t}d\nu_i^-(t) - \int_0^\varepsilon e^{-\alpha t}d\nu_i^-(t;\varepsilon)\right); \tau > \varepsilon\right\}\right|$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

$$\tag{3.20}$$

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The term under expectation in I_1 is majorized by $\int_0^\infty e^{-\alpha t} h(y_x(t)) dt$ which has finite mean by assumption. Thus $I_1 \to 0$ as $\epsilon \to 0$ by virtue of the dominated convergence theorem. Since $y_x(t; \epsilon) = \tilde{y}_x(t)$ on $\{t < \tau\}$, we have

$$|y_x(t; \epsilon)| = |\tilde{y}_x(t)| \leq |x| + r + 1 \quad \text{on} \quad \{t \leq \epsilon < \tau\}.$$

Therefore I_2 does not exceed $E\{M\epsilon; \tau > \epsilon\} \leq M\epsilon$ where $M = \max_{|y| \leq |x| + r + 1} h(y)$.

By assumption $J_x(\nu) < \infty$ therefore, by virtue of (3.19) and (3.16) and the dominated convergence theorem, $I_3 \to 0$. By virtue of (3.18)

$$|y_x(\tau; \epsilon)| \leq |x| + r + 1 + (1 + |x| + r + 1)c_2/c_1 \tag{3.21}$$

In view of (3.6) and the strong Markov property for $y_x(\cdot; \epsilon)$, we get that I_4 does not exceed $E\{M\epsilon + c_2(1 + |y_x(\tau; \epsilon)|); \tau \leq \epsilon\}$. Therefore (3.19) and (3.21) imply $I_4 \to 0$ as $\epsilon \to 0$. On the set $\{\tau > \epsilon\}$ the functional $\nu_i^+(t), t < \epsilon$ is bounded by $|x| + r + 1$. Likewise for $\nu_i^+(t; \epsilon)$. Straightforward verification shows that both $\int_0^\epsilon e^{-\alpha t} d\nu_i^+(t)$ and $\int_0^\epsilon e^{-\alpha t} d\nu_i^+(t; \epsilon)$ converge to $\nu(0)$ as $\epsilon \to 0$. Therefore by the bounded convergence theorem $I_5 \to 0$ as $\epsilon \to 0$. Similarly for I_6.

Let $V' = \bigcup_{\epsilon > 0} V_\epsilon$.

(3.22) <u>Theorem</u> The set V' is dense in V_0 that is

$$\inf_{\nu \in V_0} J_x(\nu) = \inf_{\nu \in V'} J_x(\nu).$$

<u>Proof</u> 1°. Let $\nu \in V$ such that $J_x(\nu) < \infty$.

Put

$$\nu_i^\pm(t, \delta) = \begin{cases} 0, & \text{if } 0 \leq t \leq \delta, \\ \delta^{-1} \int_{t-\delta}^t \min(\nu_i^\pm(s), \delta^{-1}) ds, & \text{if } t \geq \delta. \end{cases}$$

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It is obvious that $\nu(t;\delta) \equiv (\nu_1^+(t,\delta) - \nu_1^-(t,\delta),\ldots,\nu_n^+(t,\delta) - \nu_n^-(t,\delta))$ is a continuous functional with

$$|\dot{\nu}(t,\delta)| \leq n\delta^{-2}.$$

It is also clear that

$$\nu_i^\pm(t;\delta) \leq \nu_i^\pm(t) \tag{3.23}$$

and $\nu(t;\delta) \to \nu(t)$ as $\delta \to 0$ for all t except possibly a countable set of the points of discontinuity of ν. To justify convergence of $J_x(\nu(\cdot,\delta))$ to $J_x(\nu)$ we, however, need several extra steps as well as a further modification of $\nu(\cdot,\delta)$.

Proof 2°. Let y_x be the trajectory associated with ν and y_x^δ be the trajectory associated with $\nu(\cdot,\delta)$ and let

$$\tau_R = \inf\{t : |y_x(t)| \geq R\},$$

$$\tau_R^\delta = \inf\{t : |y_x^\delta(t)| \geq R\}.$$

Fix R. By virtue of the Theorem (3.10) there exist $\epsilon > 0$ and $\eta_x(t)\epsilon V_\epsilon$ such that

$$J_x(\eta_x) < c(1 + |x|) \quad \text{for all} \quad |x| < 4nR. \tag{3.24}$$

For every δ less than ϵ above, put

$$\sigma(R) = \tau_R \wedge \tau_{4nR}^\delta \wedge \inf\{t : \max_i \nu_i^+(t) \vee \max_i \nu_i^-(t) = N\}$$

(the constant N will be chosen later) and put

$$\bar{\nu}(t,\delta) = \begin{cases} \nu(t,\delta), & \text{if } t < \sigma(R), \\ \nu(\sigma(R),\delta) + \eta_{y_{\sigma(R)}}^\delta(t - \sigma(R)), & \text{if } t \geq \sigma(R). \end{cases}$$

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The policy $\mathcal{D}(t, \delta)$ coincides with policy $\nu(t, \delta)$ which approximates the original policy ν until either the process $|y_x(t)|$ reaches R or the process $|y_x^\delta(t)|$ reaches the level $4nR$ or one of the control functionals $\nu_i^\pm(\cdot)$ exceeds N. After that $\mathcal{D}(\cdot, \delta)$ switches to the policy whose expected cost grows with x at the rate not exceeding $|x|$. It is obvious that $\nu(\cdot, \delta) \epsilon V_\delta$. In the sections $3°$ and $4°$ we will show

$$\varlimsup_{\substack{\delta \to 0 \\ R \to \infty}} J_x(\mathcal{D}(\cdot, \delta)) \leq J_x(\nu). \tag{3.25}$$

$3°$. From the definition of $\nu(\cdot, \delta)$ it is clear that

$$\nu_i^\pm(t-) \leq \varlimsup_{\delta \to 0} \nu_i^\pm(t, \delta) \leq \nu_i^\pm(t).$$

Similarly if $t_n \to t$

$$\nu_i^\pm(t-) \leq \varlimsup_{\delta \to 0, t_n \to t} \nu^\pm(t_n, \delta) \leq \nu_i^\pm(t). \tag{3.26}$$

Suppose that with positive probability

$$\varliminf_{\delta \to 0} \tau_{4nR}^\delta < \tau_R. \tag{3.27}$$

Let $w(t)$ be the trajectory for which (3.27) holds. Then there exists a sequence $\delta_n \downarrow 0$ and a bounded sequence $t_n < \tau_R$ such that

$$|y_{xi}^{\delta_n}(t_n)| \geq 4R. \tag{3.28}$$

By choosing a subsequence if necessary, we may assume $t_n \to t$. Since $|y_{xi}(t_n)| < R$ on $\{t_n < \tau_R\}$, (3.28)) implies

$$\lim_{n \to \infty} |y_{xi}^{\delta_n}(t_n) - y_{xi}(t_n)| = \lim_{n \to \infty} |\nu_i(t_n, \delta_n) - \nu_i(t_n)| \geq 3R. \tag{3.29}$$

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However ν_i is continuous from the right and has left limits. Therefore

$$\nu_i^{\pm}(t-) \leq \overline{\lim_{t_n \to t}} \nu_i^{\pm}(t_n) \leq \nu_i^{\pm}(t). \qquad (3.30)$$

Inequality (3.30) and (3.26) show that

$$\underline{\lim}|\nu_i(t_n, \delta_n) - \nu_i(t_n)| \leq |\nu_i(t) - \nu_i(t-)| \qquad (3.31)$$

however, $|\nu_i(t) - \nu_i(t-)| = |y_{xi}(t) - y_{xi}(t-)| \leq 2R$ on the set $\{t < \tau_R\}$. The latter contradicts to (3.31) and (3.29). Therefore the probability of (3.27) is null.

$4°$. For any policy ν put

$$j_x(t, \nu) = \int_0^t e^{-\alpha t} h(y_x(t)) dt + \sum_{i=1}^n \left[a_i \int_0^t e^{-\alpha t} d\nu_i^+(t) + b_i \int_0^t e^{-\alpha t} d\nu_i^-(t) \right].$$

Since $\tau_R \uparrow \infty$ as $R \to \infty$, we can apply the dominated convergence theorem to obtain

$$J_x(\nu) = \lim_{R \to \infty} E\{j_x(\tau_R, \nu)\},$$

which implies

$$\lim_{R \to \infty} E\{j_x(\infty, \nu) - j_x(\tau_R, \nu)\} = 0.$$

Applying the strong Markov property for $y_x(\cdot)$ and the first part of the inequality (3.4), we get

$$E\{e^{-\tau_R} cR\} \leq E\{e^{-\tau_R} c|y_x(\tau_R)|\} \to 0 \quad \text{as} \quad R \to \infty. \qquad (3.32)$$

Consider

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$$J_x(\varPi(\cdot, \delta)) = E\{j_x(\infty, \varPi(\cdot, \delta))\}$$

$$(3.33)$$

$$= E\{j_x(\sigma(R), \varPi(\cdot, \delta))\} + E\{j_x(\infty, \varPi(\cdot, \delta)) - j_x(\sigma(R), \varPi(\cdot, \delta))\}$$

In view of (3.32) there exists R such that

$$c(1 + 4nR)E\{e^{-\tau_R}\} < \epsilon. \qquad (3.34)$$

Then choose N such that

$$c(1 + 4nR)P\{\max_i \nu_i^+(\tau_R) \vee \max_i \nu_i^-(\tau_R) > N\} < \epsilon. \qquad (3.35)$$

since (3.27) does not hold a.s.

$$\sigma(R) \to \tau_R \wedge \inf\{t : \max \nu_i^+(t) \vee \max \nu_i^-(t) = N\} \equiv \varsigma, \text{ as } \delta \to 0.$$

Since $\varPi(t, \delta) \to \nu(t)$ for all $t < \varsigma$ except a countable number of t

$$j_x(\sigma(R), \varPi(\cdot, \delta)) \to j_x(\varsigma, \nu). \qquad (3.36)$$

Moreover the convergence in (3.36) is bounded because $|y_x(t)| < R$ and $|y_x^\delta(t)| < 4R$ and $\nu_i^\pm(t; \delta) \leq \nu_i^\pm(t) \leq N$ if $t < \sigma(R)$. Therefore

$$\lim_{\delta \to 0} E\{j_x(\sigma(R), \varPi(\cdot, \delta)\} \to E\{j_x(\varsigma, \nu)\} \leq J_x(\nu). \qquad (3.37)$$

In view of (3.24) and the strong Markov property for y_x^δ the second term in (3.33) does not exceed

$$E\{e^{-\alpha\sigma(R)}c(1 + |y_x^\delta(\sigma(R))|)\}$$

$$\leq c(1 + 4R)E\{e^{-\alpha\sigma(R)}\}$$

$$\leq c(1 + 4R)\Big[E\{e^{-\alpha\tau_R}; \tau_R = \sigma(R)\}$$

$$+E\{e^{-\alpha\sigma(R)}; \tau_R > \tau_{4R}^\delta\} \tag{3.38}$$

$$+E\{e^{-\alpha\sigma(R)}; \tau_R > \varsigma\}\Big]$$

$$\leq c(1 + 4R)E\{e^{-\alpha\tau_R}\} + c(1 + 4R)P\{\tau_R > \tau_{4R}^\delta\}$$

$$+c(1 + 4R)P\{\tau_R > \varsigma\}.$$

The first term in (3.38) does not exceed ε by virtue of (3.34). Since (3.27) holds with probability zero the second term in (3.38) can be made smaller than ε when δ is sufficiently small. The third term in (3.38) does not exceed ε in view of (3.35). In view of arbitrariness of ε, we get (3.25). The statement of the theorem is a trivial consequence of (3.25).

(3.39) <u>Corollary.</u> For every x

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u(x).$$

The proof of this theorem follows from Proposition (3.15) and Theorems (3.22).

(3.40) <u>Theorem.</u> The optimal cost u given by (1.5) satisfies (1.11).

Proof. The proof of Theorem (3.10) shows that there exists a function u_0 for which (3.11) and (3.12) hold.

Since $\beta(q)$ is a continuous function of q and $\nabla u_{\varepsilon_k} \to \nabla u$ we have

$$\beta(\nabla u_0) = 0 \tag{3.41}$$

and in view of (1.10) we get (3.2). From (2.6)

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$$Au_\epsilon \le au_\epsilon + \epsilon^{-1}\beta(\nabla u_\epsilon) = h$$

and it follows (after passing to a limit as $\epsilon_k \to 0$)

$$Au_0 \le h$$

Suppose $\gamma(\nabla u_0(x_0)) < 1$, that is

$$-a_i < \nabla u_0(x_0)_i < b_i \quad \text{for all} \quad 1 \le i \le n. \tag{3.42}$$

By virtue of the continuity of ∇u_0, the inequality (3.42) is true for all $|x - x_0| < \delta$ for some $\delta > 0$ since convergence of ∇u_{ϵ_k} to ∇u_0 is locally uniform (see the proof of Theorem (3.10)) the inequality (3.42) is true for ∇u_{ϵ_k} for all $|x - x_0| < \delta$ and all k sufficiently large. For such k

$$Au_{\epsilon_k} = h$$

and, passing to a limit as $k \to \infty$,

$$Au_0 = h, \tag{3.43}$$

that is $\gamma(\nabla u) < 1$ implies (3.43). In view of corollary (3.39), $u = \lim u_\epsilon$, hence $u_0 = u$ and the theorem is proved.

4. Construction of the optimal policy

Let

$$\triangle_2 h(x, y) = (h(x) + h(y))/2 - h((x + y)/2). \tag{4.1}$$

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By virtue of (2.3)

$$\triangle_2 h(x,y) > 0, \quad \text{if} \quad x \neq y. \tag{4.2}$$

(4.3) <u>Theorem.</u> The optimal policy ν^* (if exists) is unique.

<u>Proof.</u> Suppose there are ν_1^* and ν_2^* for which (1.6) is true. Let $y_x^1(t)$ and $y_x^2(t)$ be the corresponding trajectories. Put $\nu = (\nu_1^* + \nu_2^*)/2$ and $y_x(t) = (y_x^1(t) + y_x^2(t))/2$. Then

$$
\begin{aligned}
u(x) - J_x(\nu) =& (J_x(\nu_1^*) + J_x(\nu_2^*))/2 - J_x(\nu) \\
\geq & E\Big\{ \int_0^\infty e^{-\alpha t} \triangle_2 h(y_x^1(t), y_x^2(t)) dt \Big\}.
\end{aligned}
\tag{4.4}
$$

By virtue of (4.2) the right hand side of (4.4) is strictly positive if ν_1^* and ν_2^* are not equal a.s.

Let m_T be a measure on $([0,T] \times \Omega, \mathcal{B}[0,T] \times \mathcal{F})$ equal to the product of Lebesgue measure and P.

(4.5) <u>Theorem.</u> If

$$J_x(\nu_k) \to u(x) \quad \text{as} \quad k \to \infty, \tag{4.6}$$

then $\nu_k(t,\omega)$ converges in measure m_T.

<u>Proof.</u> 1°. Let y_x^k be the trajectory corresponding to ν_k. then

$$E\Big\{ \int_0^T 1_{|y_x^k(t)>N|} dt \Big\} \to 0 \quad \text{as} \quad N \to \infty \tag{4.7}$$

uniformly in k. Really

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$$J_x(\nu_k) \geq e^{-\alpha T} E\left\{ \int_0^T h(y_x^k(t))dt \right\}$$

$$\geq e^{-\alpha T} \inf_{|x| \geq N} h(x) E\left\{ \int_0^T 1_{|y_x^k(t)| > N} dt \right\}. \tag{4.8}$$

Because $J_x(\nu_k)$ is uniformly bounded in k and $\lim_{N \to \infty} \inf_{|x| \geq N} h(x) = \infty$, we get (4.7).

2°. We need to show that for any $\epsilon > 0$

$$E\left\{ \int_0^T 1_{|y_x^k(t) - y_x^m(t)| > \epsilon} dt \right\} \to 0 \quad \text{as} \quad m, k \to \infty. \tag{4.9}$$

Suppose that the expectation in (4.9) is greater than $\delta > 0$ for all m and k (or for all m and k from a subsequence). Let N be such that the expectation in (4.7) is less than $\delta/2$ for all k. Then

$$(J_x(\nu_k) + J_x(\nu_m))/2 - J_x((\nu_k + \nu_m)/2)$$

$$\geq e^{-\alpha T} E\left\{ \int_0^T \triangle_2 h(y_x^m(t), y_x^k(t)) 1_{|y_x^k(t)| \leq N} 1_{|y_x^m(t)| \leq N} \right\} dt$$

$$\geq e^{-\alpha T} \rho(\epsilon, N) E\left\{ \int_0^T 1_{|y_x^k(t) - y_x^m(t)| > \epsilon} 1_{|y_x^m(t)| \leq N} 1_{|y_x^m(t)| \leq N} dt \right\}$$

$$e^{-\alpha T} \rho(\epsilon, N) \delta/2, \tag{4.10}$$

where $\rho(\epsilon, N) = \inf_{|x-y| \geq \epsilon; |x|, |y| \leq N} \triangle_2 h(x, y)$. (Usual continuity/compactness arguments show that $\rho(\epsilon, N) > 0$ for any N). Inequality (4.10) and (4.6) imply $\underline{\lim} J_x((\nu_k + \nu_m)/2) < u(x)$ and we come to a contradiction.

(4.11) <u>Corollary.</u> There exists an optimal policy ν^*.

<u>Proof.</u> Taking a sequence ν_k for which (4.5) is true, and using a diagonal method, we can find a subsequence ν_{k_m} which converges a.e. m_T for each $T > 0$. Usual argument show an existence of ν such that

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$$\nu(t,\omega) = \lim \nu_{k_m}(t,\omega) \qquad (4.12)$$

for $Leb \times P$ almost all (t,ω). By Fatou's lemma

$$\lim J_x(\nu_{k_m}) \geq J_x(\lim(\nu_{k_m})) = J_x(\nu).$$

Thus $J_x(\nu) \leq u(x)$. Therefore ν given by (4.12) coincides with ν^*.

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